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On the Well-Posedness of the Inverse Sturm–Liouville Problems

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INTRODUCTION

In previous publications [1, 2], the author considered the following problems. Let $(L_p\,; \alpha, \beta)$ denote the differential operator

$$L_p u = -u'' + q(x)\, u \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} u(0) \cos \alpha + u'(0) \sin \alpha &= 0, \\ u(1) \cos \beta + u'(1) \sin \beta &= 0. \end{aligned} \tag{2}$$

$q(x)$ is assumed to be integrable on $[0, 1]$. Suppose a second operator $(\tilde{L}_p\,; \alpha, \beta)$ is defined by

$$\tilde{L}_p u = -u'' + \tilde{q}(x)\, u \tag{3}$$

and the boundary conditions (2). We shall denote the spectrum of $(L_p\,; \alpha, \beta)$ by $\{\lambda_i\}$ and that of $(\tilde{L}_p\,; \alpha, \beta)$ by $\{\tilde{\lambda}_i\}$. Suppose now that $(L_p\,; \alpha, \gamma)$ has the spectrum $\{\lambda_i'\}$ and $(\tilde{L}_p\,; \alpha, \gamma)$ has the spectrum $\{\tilde{\lambda}_i'\}$, where $\sin(\beta - \gamma) \neq 0$.

In [1] the following result was proved. Suppose $\lambda_i' = \tilde{\lambda}_i'$ for all $i = 1, 2,\ldots$, but that $\lambda_i = \tilde{\lambda}_i$, $i \in \Lambda$, $\lambda_i \neq \tilde{\lambda}_i$, $i \in \Lambda_0$. Here Λ_0 is a finite index set and Λ an infinite index set and $\Lambda_0 \cup \Lambda = \{1, 2,\ldots\}$. Under these conditions

$$q(x) - \tilde{q}(x) = \sum_{i \in \Lambda_0} (\tilde{y}_i w_i)' \qquad \text{(almost everywhere)}, \tag{4}$$

where $\tilde{y}_i$ is a suitable solution (to be defined in detail later) of

$$\tilde{y}_i'' + [\lambda_i - \tilde{q}(x)]\, \tilde{y}_i = 0 \tag{5}$$

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and

$$w_i'' + [\lambda_i - q(x)]w_i = 0,$$
$$w_i(0) = \sin\alpha, \qquad w_i'(0) = -\cos\alpha, \tag{6}$$

N.B. $\tilde{y}_i$ is not an eigenfunction of $\tilde{L}_p$; $\alpha, \beta)$ since λ_i is not in the spectrum of that operator for $i \in \Lambda_0$. But w_i is an eigenfunction of $(L_p ; \alpha, \beta)$.

Formula (4) shows a relationship between two potential functions $q(x)$ and $\tilde{q}(x)$ for which the above operators have such intimately related spectra. In particular if Λ_0 is empty (4) shows that $q(x) - \tilde{q}(x) = 0$ (almost everywhere). In other words the spectra of $(L_p ; \alpha, \beta)$ and $(L_p ; \alpha, \gamma)$ uniquely (a.e.) determine $q(x)$. This raises the question whether this inverse spectral problem is well posed. It will be shown that if

$$|\lambda_i - \tilde{\lambda}_i| < \epsilon, \qquad i \in \Lambda_0, \tag{7}$$

then

$$|q(x) - \tilde{q}(x)| < K\epsilon \qquad \text{(a.e.)}. \tag{8}$$

In other words perturbations in a finite set of eigenvalues of $(L_p ; \alpha, \beta)$ lead to small perturbations in $q(x)$.

A second class of inverse problems concern themselves with operators of the form

$$L_d u = [1/\rho(x)]\, u'' \tag{9}$$

subject to the boundary conditions (2). We shall denote the above operator by $(L_d ; \alpha, \beta)$. The subscript p in (1) refers to the fact that such operators arise in quantum mechanics, where $q(x)$ is referred to as a potential. The operator in (9) arises, for example, in the study of vibrating strings. There $\rho(x)$ is the density function of the string and the operator is given the subscript d. For this class of operators the well-posedness of the inverse spectral problem will also be proved.

Statement of Results

Theorem 1. *Let $(L_p ; \alpha, \beta)$ be the operator defined by* (1) *and* (2) *and denote its spectrum by $\{\lambda_i\}$. Let $(L_p ; \alpha, \gamma)$ be similarly defined with* $\sin(\beta - \gamma) \neq 0$ *and denote its spectrum by $\{\lambda_i'\}$. Similarly $(\tilde{L}_p ; \beta)$ with spectrum $\{\tilde{\lambda}_i\}$ and $(\tilde{L}_p ; \alpha, \gamma)$ with spectrum $\{\tilde{\lambda}_i'\}$ can be defined as in* (3). *We assume that q and $\tilde{q}$ are summable on* $[0, 1]$.

We suppose that $\lambda_i' = \tilde{\lambda}_i'$ for $i = 1, 2, \ldots$ and that $\lambda_i = \tilde{\lambda}_i$ for $i \in \Lambda$, but that $\lambda_i \neq \tilde{\lambda}_i$ for $i \in \Lambda_0$. Here Λ is an infinite index set and Λ_0 a finite index set and $\Lambda_0 \cup \Lambda = \{1, 2, \ldots\}$. If

$$|\lambda_i - \tilde{\lambda}_i| < \epsilon \qquad \textit{for} \quad i \in \Lambda_0 \tag{10}$$

then

$$| q(x) - \tilde{q}(x)| < K\epsilon \qquad (a.e.) \tag{11}$$

for a suitable constant K.

THEOREM 2. *Consider $(L_p\,; \alpha, -\alpha)$ and $(\tilde{L}_p\,; \alpha, -\alpha)$ where the potential functions q and $\tilde{q}$ satisfy*

$$q(x) = q(1 - x), \qquad \tilde{q}(x) = \tilde{q}(1 - x), \tag{12}$$

and are summable on $[0, 1]$.

Let the spectra of the above operators be $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$. If (as in Theorem 1) $\lambda_i = \tilde{\lambda}_i$, $i \in \Lambda$ and $\lambda_i \neq \tilde{\lambda}_i$, $i \in \Lambda_0$ and (10) holds than (11) again follows.

THEOREM 3. *Let $(L_d\,; \alpha, \beta)$ and $(L_d\,; \alpha, \gamma)$ be defined by* (9) *and* (2) *and their spectra be denoted by $\{\lambda_i\}$ and $\{\lambda_i'\}$. Let $(\tilde{L}_d\,; \alpha, \beta)$ and $(\tilde{L}_d\,; \alpha, \gamma)$ be defined by* (9) *and* (2) *with $\rho(x)$ replaced by $\tilde{\rho}(x)$ and we assume that* $\sin(\beta - \gamma) \neq 0$. *The spectra of the last two operators shall be denoted by $\{\tilde{\lambda}_i\}$ and $\{\tilde{\lambda}_i'\}$, respectively. We shall assume that $\rho(x)$ and $\tilde{\rho}(x)$ are twice differentiable and positive on* $[0, 1]$, *and that* $\rho(0)/\rho(1) = \tilde{\rho}(0)/\tilde{\rho}(1)$. *Suppose $\lambda_i' = \tilde{\lambda}_i'$ for $i = 1, 2,...$ and $\lambda_i = \tilde{\lambda}_i$, $i \in \Lambda$ and $\lambda_i \neq \tilde{\lambda}_i$, $i \in \Lambda_0$, where Λ is an infinite index set and Λ_0 is a finite index set and $\Lambda_0 \cup \Lambda = \{1, 2,...\}$. If*

$$| \lambda_i - \tilde{\lambda}_i | < \epsilon, \qquad i \in \Lambda_0 , \tag{13}$$

then

$$| \rho(x) - \tilde{\rho}(x)| < K\epsilon \tag{14}$$

for a suitable constant K.

THEOREM 4. *Consider $(L_d\,; \alpha, -\alpha)$ and $(\tilde{L}_d\,; \alpha, -\alpha)$ where the density functions $\rho(x)$ and $\tilde{\rho}(x)$ are twice differentiable and positive on* $[0, 1]$ *and satisfy*

$$\rho(x) = \rho(1 - x); \qquad \tilde{\rho}(x) = \tilde{\rho}(1 - x). \tag{15}$$

Let the spectra of these operators be $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$. If (as in Theorem 3) *$\lambda_i = \tilde{\lambda}_i$, $i \in \Lambda$ and $\lambda_i \neq \tilde{\lambda}_i$, $i \in \Lambda_0$ and* (13) *holds then* (14) *follows.*

In a final section these results will be applied to calculate small changes in potentials and densities resulting from small changes in the spectrum.

PROOFS OF THE THEOREMS

In order to accomplish the proofs of the theorems free use will be made of all the results to be found in [1, 2].

Proof of Theorem 1. In [1] it was shown that under the hypotheses of Theorem 1,

$$q - \tilde{q} = \sum_{i \in \Lambda_0} (\tilde{y}_i w_i)' \tag{16}$$

where

$$\begin{gathered} w_i'' + [\lambda_i - q(x)] w_i = 0, \\ w_i(0) = \sin\alpha, \qquad w_i'(0) = -\cos\alpha, \end{gathered} \tag{17}$$

and

$$\tilde{y}_i = 2(\tilde{u}_i - k_i \tilde{z}_i)/\omega'(\lambda_i). \tag{18}$$

In the above $\tilde{u}_i$ and $\tilde{z}_i$ are solutions of the equation

$$u_i'' + [\lambda_i - \tilde{q}(x)] u_i = 0 \tag{19}$$

with the initial conditions

$$\tilde{z}_i(0) = \quad \sin\alpha, \qquad \tilde{z}_i'(0) = -\cos\alpha, \tag{20}$$

$$\tilde{u}_i(1) = -\sin\beta, \qquad \tilde{u}_i'(1) = \quad \cos\beta. \tag{21}$$

The function $\omega(\lambda)$ is defined by

$$\omega(\lambda) = w(1)\cos\beta + w'(1)\sin\beta, \tag{22}$$

where w is a solution of (17) with λ_i replaced by λ. It is an entire function of λ of order one-half and its zeros are the eigenvalues of $(L_p\,; \alpha, \beta)$. The scalar k_i is given by

$$k_i = [\sin(\gamma - \beta)/\nu(\lambda_i)] \tag{23}$$

where

$$\nu(\lambda) = w(1)\cos\gamma + w'(1)\sin\gamma \tag{24}$$

is an entire function of λ of order one-half. Its zeros are given by the spectrum of $(L_p\,; \alpha, \gamma)$ and it is uniquely defined by these zeros and its asymptotic form.

From (10) we note that $\lambda_i = \tilde{\lambda}_i + \epsilon\zeta_i$, where $|\zeta_i| < 1$. Then (19) can be written in the form

$$u_i'' + [\tilde{\lambda}_i - \tilde{q}(x)] u_i = -\epsilon\zeta_i u_i\,. \tag{25}$$

First we shall examine the solution $\tilde{z}_i$ satisfying (20). Since $\tilde{\lambda}_i$ is in the spectrum of $(\tilde{L}_p\,; \alpha, \beta)$ $\tilde{z}_i$ must be close to an eigenfunction of that operator. For $\zeta_i = 0$ $\tilde{z}_i$ would indeed be an eigenfunction, say $\tilde{w}_i$ (just as w_i defined in (17) is an eigenfunction of $(L_p\,; \alpha, \beta)$). This eigenfunction is so normalized that it satisfies suitable initial conditions. From standard results in perturbation theory one obtains from (25)

$$\tilde{z}_i = \tilde{w}_i + \epsilon \tilde{r}_i . \tag{26}$$

Here $\tilde{r}_i$ is a twice differentiable function that may depend on ϵ. Similarly

$$\tilde{u}_i = \tilde{v}_i + \epsilon \tilde{s}_i, \tag{27}$$

where $\tilde{v}_i$ is an eigenfunction of $(\tilde{L}_p\,;\alpha,\beta)$ satisfying the end point conditions (21).

Now $\tilde{v}_i$ and $\tilde{w}_i$ are differently normalized eigenfunctions of $(\tilde{L}_p\,;\alpha,\beta)$ and hence linearly dependent. Then

$$\tilde{v}_i = \tilde{k}_i \tilde{w}_i, \tag{28}$$

where (see[1])

$$\tilde{k}_i = [\sin(\gamma - \beta)/\tilde{\nu}(\tilde{\lambda}_i)]. \tag{29}$$

Since the spectra of $(L_p\,;\alpha,\gamma)$ and $(\tilde{L}_p\,;\alpha,\gamma)$ coincide $\nu(\lambda) = \tilde{\nu}(\lambda)$ (see (24)) and hence

$$| \nu(\lambda_i) - \nu(\tilde{\lambda}_i) | = | \nu'(\theta_i) | \, | \lambda_i - \tilde{\lambda}_i | < K\epsilon, \tag{30}$$

for a suitable constant K and θ_i lies between λ_i and $\tilde{\lambda}_i$. It follows that

$$k_i = \tilde{k}_i + \epsilon l_i \tag{31}$$

for a suitable scalar l_i. Inserting (26), (27), and (31) in (18) (and using (28)) shows that

$$| \tilde{y}_i | < K\epsilon, \tag{32}$$

and similarly

$$| \tilde{y}_i' | < K\epsilon, \tag{33}$$

for suitable K. Finally, using (16) we have

$$| q - \tilde{q} | < K\epsilon, \tag{34}$$

for some K.

Proof of Theorem 2. Steps (16), (18), (26), and (27) are as in the previous proof. The chief difference lies in the computation of k_i for which the second spectrum was required. In the underlying symmetric case only one spectrum is used. The results of [1] show that in this case

$$k_i = \operatorname{sgn} \omega'(\lambda_i), \tag{35}$$

$$\tilde{k}_i = \operatorname{sgn} \omega'(\lambda_i), \tag{36}$$

so that

$$k_i = \tilde{k}_i\,, \tag{37}$$

which replaces (31). Thus the conclusion (34) follows as before.

Proof of Theorem 3. In conjunction with the density functions $\rho(x)$ and $\tilde{\rho}(x)$ two other functions can be introduced.

$$\xi(x) = \int_0^x \rho(t)^{1/2}\,dt, \qquad \tilde{\xi}(x) = \int_0^x \tilde{\rho}(t)^{1/2}\,dt. \tag{38}$$

Since both densities are positive, ξ and $\tilde{\xi}$ are monotonically increasing. We also know that (see [2])

$$\lim_{n\to\infty} (\lambda_n^{1/2}/n) = \pi/\xi(1),$$

and that $\lambda_n = \tilde{\lambda}_n$ for n sufficiently large. Thus $\xi(1) = \tilde{\xi}(1)$, and of course $\xi(0) = \tilde{\xi}(0)$. It follows that there exists a unique function $\tilde{x}(x)$ satisfying the properties

$$\begin{gathered} \tilde{\xi}(x) = \xi(\tilde{x}), \\ \tilde{x}(0) = 0, \qquad \tilde{x}(1) = 1, \end{gathered} \tag{39}$$

and by using (38)

$$\tilde{x}'(x) = (\tilde{\rho}(x)/\rho(\tilde{x}))^{1/2}. \tag{40}$$

It is also convenient to define $\tau(x)$ as

$$\tau(x) = (\rho(\tilde{x})/\tilde{\rho}(x))^{1/4} = (\tilde{x}'(x))^{-1/2}. \tag{41}$$

In [2] it was shown that, under the hypotheses of the theorem,

$$\begin{aligned} \tau''(x) - \sum_{n\in A_0} (&2\tilde{y}_n'(x)\,\rho(\tilde{x})\,w_n(\tilde{x})\,\tilde{x}'(x) + 2\tilde{\rho}(x)\,\tilde{y}_n(x)\,w_n'(\tilde{x}) \\ &+ \tilde{y}_n(x)\,\rho'(\tilde{x})\,w_n(\tilde{x})\,\tilde{x}'^2(x) + \tilde{y}_n(x)\,\rho(\tilde{x})\,w_n(\tilde{x})\,\tilde{x}''(x)) = 0, \end{aligned} \tag{42}$$

by analogy to (16) in the proof of Theorem 1.

$w_n(x)$ in (42) is defined as the solution of the initial value problem

$$\begin{gathered} w_n'' + \lambda_n \rho(x) w_n = 0, \\ w_n(0) = \sin\alpha, \qquad w_n'(0) = -\cos\alpha. \end{gathered} \tag{43}$$

Since λ_n is in the spectrum of $(L_d\,;\alpha,\beta)$ $w_n(x)$ is an eigenfunction of that problem. Finally

$$\tilde{y}_n(x) = [\tilde{u}_n(x) - k_n\tilde{z}_n(x)]/\omega'(\lambda_n), \tag{44}$$

where $\tilde{u}_n$ and $\tilde{z}_n$ are solutions of

$$u_n'' + \lambda_n\tilde{\rho}(x)u_n = 0 \tag{45}$$

satisfying the endpoint conditions

$$\tilde{z}_n(0) = \sin\alpha, \qquad \tilde{z}_n'(0) = -\cos\alpha, \tag{46}$$

$$\tilde{u}_n(1) = -\sin\beta, \qquad \tilde{u}_n'(1) = \cos\beta, \tag{47}$$

The function $\omega(\lambda)$ is defined exactly as in (22), and the constant k_n as in (23). From this point on the proof is very similar to that of Theorem 1. Let $\lambda_n = \tilde{\lambda}_n + \epsilon\zeta_n$, where $|\zeta_n| < 1$. Then (45) reduces to

$$u_n'' + \tilde{\lambda}_n\tilde{\rho}(x)u_n = -\epsilon\zeta_n\tilde{\rho}(x)u_n\,. \tag{48}$$

$\tilde{\lambda}_n$ is in the spectrum of $(L_d\,;\alpha,\beta)$ so that for $\epsilon = 0$ u_n would be an eigenfunction. Then by perturbation results

$$\tilde{z}_n = \tilde{w}_n + \epsilon\tilde{r}_n, \tag{49}$$

$$\tilde{u}_n = \tilde{v}_n + \epsilon\tilde{s}_n. \tag{50}$$

Here $\tilde{w}_n$ and $\tilde{v}_n$ are linearly dependent eigenfunctions so that (see [2])

$$\tilde{v}_n = \tilde{k}_n\tilde{w}_n \tag{51}$$

where

$$\tilde{k}_n = [\sin(\nu - \beta)/\tilde{\nu}(\tilde{\lambda}_n)]. \tag{52}$$

Since the spectra of $(L_d\,;\alpha,\gamma)$ and $(\tilde{L}_d\,;\alpha,\beta)$ are identical,

$$\nu(\lambda) = \tilde{\nu}(\lambda),$$

and also

$$|\lambda_n - \tilde{\lambda}_n| < \epsilon,$$

we have

$$|\nu(\lambda_n) - \tilde{\nu}(\tilde{\lambda}_n)| < K\epsilon,$$

and as in (32) and (33) we can conclude that

$$|\tilde{y}_n| < K\epsilon, \qquad |\tilde{y}_n'| < K\epsilon \tag{53}$$

for a suitable constant K.

From (42) we now see that

$$\tau''(x) = O(\epsilon) \tag{54}$$

so that

$$\tau(x) = \tau(0) + \tau'(0)\, x + O(\epsilon). \tag{55}$$

But by hypothesis $\tau(0) = \tau(1) = b$ (say) and then

$$\tau(x) = b + O(\epsilon). \tag{56}$$

Using (41) we see that

$$\tilde{x}(x) = (x/b^2) + O(\epsilon),$$

and since $\tilde{x}(1) = 1$ we have $b = 1$, so that

$$\tilde{x}(x) = x + O(\epsilon). \tag{57}$$

Then (41) yields

$$\rho(\tilde{x}) = \rho(x + O(\epsilon)) = \tilde{\rho}(x) + O(\epsilon),$$

and since $\rho(x)$ is twice differentiable

$$| \rho(x) - \tilde{\rho}(x)| < K\epsilon \tag{58}$$

for a suitable constant K.

Proof of Theorem 4. As is the case with Theorems 1 and 2, the proof of Theorem 4 is completely analogous to that of Theorem 3, except for the calculation of k_n. Here again (35) and (36) apply (see [2]) so that $k_n = \tilde{k}_n$ and (53) holds as before. From this step the rest of the proof of Theorem 3 applies and the conclusion follows.

Applications

To illustrate how the above results can be used to calculate potentials and densities we shall consider some particular cases. We shall suppose $\tilde{q}$ is known and that the spectrum of $(\tilde{L}_p\,; 0, 0)$ is also known. To keep matters simple we shall suppose that $\tilde{q}(x) = \tilde{q}(1 - x)$.

Now we shall suppose that the spectrum of $(L_p\,; 0, 0)$ is prescribed in such a way that

$$\begin{aligned} &\lambda_i = \tilde{\lambda}_i + \epsilon\zeta_i\,, \qquad |\zeta_i| < 1, \qquad i = 1, 2, \ldots, n, \\ &\lambda_i = \tilde{\lambda}_i\,, \qquad i > n. \end{aligned} \tag{59}$$

Then the potential $q(x)$ associated with $(L_p\,; 0, 0)$ is given by (see (16)f.f.)

$$q = \tilde{q} + \sum_{i=1}^{n} (\tilde{y}_i w_i)'. \tag{60}$$

To calculate $\tilde{y}_i$ we see that

$$\tilde{y}_i = 2(\tilde{u}_i - k_i \tilde{z}_i)/\omega'(\lambda_i).$$

Since $\tilde{\omega}(\lambda)$ is known we have

$$\omega(\lambda) = \prod_1^n [(\lambda - \lambda_i)/(\lambda - \tilde{\lambda}_i)]\, \tilde{\omega}(\lambda)$$

and

$$\omega'(\lambda_j) = \left[\prod_{1, i\neq j}^{n} (\lambda_j - \lambda_i) \Big/ \prod_{i=1}^{n} (\lambda_j - \tilde{\lambda}_i)\right] \tilde{\omega}(\lambda_j) = [\tilde{\omega}(\tilde{\lambda}_j + \epsilon\zeta_j)/\epsilon\zeta_j] + O(\epsilon)$$

$$= \tilde{\omega}'(\tilde{\lambda}_j) + O(\epsilon) \tag{61}$$

since $\tilde{\omega}(\tilde{\lambda}_j) = 0$.
By (35)

$$k_j = \operatorname{sgn} \omega'(\lambda_j) = (-1)^j. \tag{62}$$

Now $\tilde{u}_i$ satisfies

$$\tilde{u}_i'' + [\tilde{\lambda}_i - \tilde{q}]u_i = -\epsilon\zeta_i u_i\,,$$
$$\tilde{u}_i(1) = 0, \qquad \tilde{u}_i'(1) = 1,$$

so that

$$\tilde{u}_i = \tilde{v}_i + \epsilon\zeta_i \tilde{v}_i \int_x^1 \tilde{v}_i^2 \int_t^x (1/\tilde{v}_i^2)\, d\tau\, dt + O(\epsilon^2) \tag{63}$$

and similarly

$$\tilde{z}_i = \tilde{w}_i - \epsilon\zeta_i \tilde{w}_i \int_0^x \tilde{w}_i^2 \int_t^x (1/\tilde{w}_i^2)\, d\tau\, dt + O(\epsilon^2). \tag{64}$$

The true eigenfunctions $\tilde{w}_i$ and $\tilde{v}_i$ are related by (see (28))

$$\tilde{v}_i = \tilde{k}_i \tilde{w}_i = (-1)^i\, \tilde{w}_i\,,$$

so that finally

$$\tilde{y}_i = \frac{(-1)^i\, 2\epsilon\zeta_i}{\tilde{\omega}'(\tilde{\lambda}_i)}\, \tilde{w}_i \left(\int_0^x \tilde{w}_i^2 \int_t^x \frac{1}{\tilde{w}_i^2}\, d\tau\, dt + \int_x^1 \tilde{w}_i^2 \int_t^x \frac{1}{\tilde{w}_i^2}\, d\tau\, dt\right) + O(\epsilon^2). \tag{65}$$

Finally, $\tilde{w}_i$ satisfies

$$w_i'' + [\lambda_i - q(x)]w_i = 0,$$
$$w_i(0) = 0, \qquad w_i'(0) = 1,$$

and since $\lambda_i = \tilde{\lambda}_i + O(\epsilon)$, $q(x) = \tilde{q}(x) + O(\epsilon)$

$$w_i = \tilde{w}_i + O(\epsilon). \tag{66}$$

By combining these results we have

$$q(x) = \tilde{q}(x) + \sum_{i=1}^{n} (\tilde{y}_i \tilde{w}_i)' + O(\epsilon^2) \tag{67}$$

where the second term on the right of (67) is now explicitly known.

In particular for the case where $\tilde{q}(x) = 0$ we find

$$q(x) = \sum_{k=1}^{n} 2\epsilon\zeta_k \cos 2k\pi x + O(\epsilon^2). \tag{68}$$

A similar calculation can be performed in the case of the problem $(L_a\,; 0, 0)$ with the symmetry condition $\rho(x) = \rho(1 - x)$. Here again we assume that

$$\begin{aligned} &\lambda_j = \tilde{\lambda}_j + \epsilon\zeta_j\,, \qquad |\zeta_j| < 1, \qquad j = 0, 1,\ldots, n,\\ &\lambda_j = \tilde{\lambda}_j\,, \qquad j > n, \end{aligned} \tag{69}$$

and we shall assume that $\tilde{\rho}(x) = 1$. As before

$$\tilde{y}_j = (\tilde{u}_j - k_j\tilde{z}_j)/\omega'(\lambda_j) \tag{70}$$

where

$$\begin{aligned} \tilde{\omega}(\lambda) &= \sin(\lambda^{1/2})/\lambda^{1/2}\\ \tilde{\omega}'(\tilde{\lambda}_j) &= \tfrac{1}{2}(-1)^j, \qquad j \geqslant 1,\\ k_j &= \operatorname{sgn} \omega'(\lambda_j) = (-1)^j,\\ \omega(\lambda_j) &= \tilde{\omega}(\tilde{\lambda}_j) + O(\epsilon). \end{aligned} \tag{71}$$

Now $\tilde{u}_j$ and $\tilde{z}_j$ are solutions of

$$u_j'' + \lambda_j u_j = 0 \tag{72}$$

satisfying the initial conditions

$$\begin{aligned} &\tilde{u}_j(1) = 0, \qquad \tilde{u}_j'(1) = 1,\\ &\tilde{z}_j(0) = 0, \qquad \tilde{z}_j'(0) = 1. \end{aligned} \tag{73}$$

From (71) we note that

$$\tilde{\lambda}_j = j^2\pi^2$$

so that

$$\lambda_j = j^2\pi^2 + \epsilon\zeta_j\,.$$

Inserting the above in (72) and solving we find that

$$\begin{aligned} \tilde{u}_j &= \left[\sin\left(j\pi + \frac{\epsilon\zeta_j}{2j\pi}\right)(x-1)\Big/\left(j\pi + \frac{\epsilon\zeta_j}{2j\pi}\right)\right] + O(\epsilon^2), \\ \tilde{z}_j &= \left[\sin\left(j\pi + \frac{\epsilon\zeta_j}{2j\pi}\right)x\Big/\left(j\pi + \frac{\epsilon\zeta_j}{2j\pi}\right)\right] + O(\epsilon^2). \end{aligned} \tag{74}$$

Combining all these we finally obtain

$$\tilde{y}_j = 2\epsilon\zeta_j \cos j\pi x + O(\epsilon^2). \tag{75}$$

Similarly, w_j satisfies

$$\begin{aligned} w_j'' + \lambda_j\rho(x)w_j &= 0, \\ w_j(0) = 0, \qquad w_j'(0) &= 1, \end{aligned} \tag{76}$$

and since

$$\rho(x) = \tilde{\rho}(x) + O(\epsilon) = 1 + O(\epsilon)$$

we find

$$w_j = (\sin j\pi x/j\pi) + O(\epsilon). \tag{77}$$

Next, Eq. (42) has to be solved for $\tau(x)$. To simplify this nonlinear differential equation we use the fact that

$$\rho(x) = 1 + O(\epsilon)$$

so that (see (40), (41))

$$\begin{aligned} \tilde{x}'(x) &= (\rho(\tilde{x}))^{-1/2} = 1 + O(\epsilon), \\ \tau(x) &= (\tilde{x}'(x))^{-1/2} = 1 + O(\epsilon), \\ \tilde{x}''(x) &= -2\tau'(x)/\tau^3(x) = -2\tau'(x) + O(\epsilon), \end{aligned} \tag{78}$$

so that (42) reduces to

$$\tau'' - \sum_{k=0}^{n} (2\tilde{y}_k'(x)\, w_k(x) + 2\tilde{y}_k(x)\, w_k'(x) - 2\tilde{y}_k(x)\, w_k(x)\tau') = O(\epsilon^2). \tag{79}$$

Now by use of (75) and (77) the above reduces to

$$\tau'' + \left(\sum_{j=1}^{n} \frac{\epsilon\zeta_j \sin 2j\pi x}{j\pi}\right)\tau' + \sum_{j=1}^{n} 2\epsilon\zeta_j \cos 2j\pi x = O(\epsilon^2). \tag{80}$$

In solving (80) two constants of integration have to be determined. There are two properties that can be used to determine them, namely,

$$\tau(x) = \tau(1 - x)$$

and (since $\xi(1) = \tilde{\xi}(1)$)

$$\int_0^1 \tilde{\rho}(x)^{1/2}\, dx = \int_0^1 \rho(x)^{1/2}\, dx = \int_0^1 \tau^2(x)\, dx = 1. \tag{81}$$

Then, finally,

$$\rho(x) = \tau^4(x) = 1 + 2\epsilon \sum_{j=1}^{n} (\zeta_j/j^2\pi^2) \cos 2j\pi x + O(\epsilon^2). \tag{82}$$

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